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Buckling deformation in axially compressed elastic-plastic cylindrical shells initiated by local axisymmetric imperfections

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SUMMARY

A solution for growth of a perturbation describing buckling deformation initiated by a local axisymmetric imperfection in an axially compressed elastic-plastic cylindrical shell is obtained in simple closed form for small time. The constitutive relations for the plastic strains are based on simple J_2 flow theory. A comparison is made with bifurcation analysis, and localization of the buckling deformation is examined.

1. Introduction

In an earlier paper [1], the author presented a rigid-plastic analysis of localized axisymmetric buckling deformation in an axially compressed cylindrical shell in which buckling was initiated by a local geometric imperfection. The analysis yielded the solution for the time-dependent radial displacement as an infinite series that converged for all time. The predicted deformation was compared with deformation observed in experiments on moderately thick shells, radius/thickness ≈ 10, of a structural aluminium alloy. Reasonably good agreement was noted in the extent of the buckling deformation along the shell axis between prediction and experiment. In the present paper, the analysis is extended to include elastic strains. A direct comparison with bifurcation theory is thus possible. Also it is found that, for small time, the infinite series representing radial displacement can be summed in simple closed form.

2. Linearized kinematic, constitutive, and equilibrium equations

As in [1], the notation and analysis of Timoshenko and Gere [2] are followed. The coordinates x, θ, z are chosen such that the x axis lies along a meridian parallel to the axis of the cylinder, and the z axis points inward along a radius. The nominal radius of the middle surface is a, the shell thickness is h, and the radial displacement positive inward is w.

The axisymmetric stress field σ_x, σ_θ is written

$$\sigma_x = -P + s_x, \quad \sigma_\theta = s_\theta$$

where P is a uniform compressive stress, and s_x, s_θ are small perturbations due to the buckling deformation. Force resultants N_x, N_θ are associated with the perturbation stresses s_x, s_θ, and

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also couple resultants M_x, M_θ. Subsequently the axial force resultant N_x is neglected as in elementary elastic analysis [2].

Linearized kinematic relations for the strains ϵ_x, ϵ_θ are introduced by putting

$$\epsilon_x = \epsilon_x^0 + e_x + z\kappa_x, \quad \epsilon_\theta = \epsilon_\theta^0 + e_\theta + z\kappa_\theta, \tag{2.1}$$

where ϵ_x^0, ϵ_θ^0 are the uniform axial, hoop strains due to P; e_x, e_θ are middle-surface strain perturbations associated with N_x, N_θ ; and κ_x, κ_θ are middle-surface curvature changes due to M_x, M_θ. Since linear strain-displacement relations are considered, the radial displacement w can be identified with the buckling deformation, and hence,

$$e_\theta = -w/a, \quad \kappa_x = -w'', \quad \kappa_\theta = -w/a^2 \tag{2.2}$$

where primes indicate differentiation with respect to x. The curvature change κ_θ is neglected subsequently compared to κ_x, as in elementary elastic analysis.

The total strains, and their components, defined in (2.1), are now split into elastic parts ϵ_x^{el}, ϵ_θ^{el} and plastic parts ϵ_x^{pl}, ϵ_θ^{pl}. The usual Hooke's law relations then give,

$$he_x^{el} = (N_x - \nu N_\theta)/E, \quad he_\theta^{el} = (N_\theta - \nu N_x)/E, \tag{2.3}$$

$$(h^3/12)\kappa_x^{el} = (M_x - \nu M_\theta)/E, \quad (h^3/12)\kappa_\theta^{el} = (M_\theta - \nu M_x)/E, \tag{2.4}$$

where E is Young's modulus and ν is Poisson's ratio. Linearized constitutive relations for the plastic strain rates $\dot{e}_x^{pl}$, $\dot{e}_\theta^{pl}$ and plastic curvature change rates $\dot{\kappa}_x^{pl}$, $\dot{\kappa}_\theta^{pl}$, in the context of the present analysis, were previously established in [1] (equations (2.4)-(2.7)), using the von Mises yield condition and associated flow rule [3]. When the dependence of the uniform axial compressive stress P on time t is taken in the form $P = P_0 e^t$, where P_0 is a constant representing compressive stress equal to or greater than the initial yield stress of the material in compression, these relations become,

$$h\dot{e}_x^{pl} = (\dot{N}_x - \dot{N}_\theta/2)/H, \tag{2.5}$$

$$h\dot{e}_\theta^{pl} = (3N_\theta - 2\dot{N}_x + \dot{N}_\theta)/(4H), \tag{2.6}$$

and

$$(h^3/12)\dot{\kappa}_x^{pl} = (\dot{M}_x - \dot{M}_\theta/2)/H, \tag{2.7}$$

$$(h^3/12)\dot{\kappa}_\theta^{pl} = (3M_\theta - 2\dot{M}_x + \dot{M}_\theta)/(4H). \tag{2.8}$$

In (2.5) – (2.8) and the following, the superposed dot denotes differentiation with respect to t, and H is the hardening modulus, assumed to be constant over the time interval that the growth of the perturbation is followed.

The equation of radial equilibrium [2]

$$M_x'' - Phw'' + N_\theta/a = 0 \tag{2.9}$$

completes the specification of field equations for the problem.

3. Bifurcation analysis

Bifurcation analysis applies to an initially perfect shell that undergoes uniform deformation. For $t \leqslant 0$, $w \equiv N_x \equiv N_\theta \equiv M_x \equiv M_\theta \equiv 0$. Then, at $t = 0$, the possibility that $\dot{w}, \dot{N}_x, \dot{N}_\theta, \dot{M}_x, \dot{M}_\theta \neq 0$ is investigated. With the additional assumptions of elementary shell theory that N_x, κ_θ are negligible, adding the elastic and plastic strain rates obtained from (2.3) – (2.8) to form total strain rates, and then using (2.2) lead to the relations for the instant $t = 0$,

$$\dot{M}_x = -\frac{Hh^3}{12}\,\frac{1+4H/E}{(5-4\nu)H/E+4(1-\nu^2)H^2/E^2}\,\dot{w}'', \tag{3.1}$$

$$\dot{N}_\theta = -\frac{4Hh}{(1+4H/E)a}\,\dot{w}. \tag{3.2}$$

It is convenient to introduce dimensionless quantities:

$$\xi = x\,(6/ah)^{1/2}, \qquad W = wh/a^2, \qquad \beta = H/E, \tag{3.3}$$

$$\psi = 3Pa/(2Hh), \qquad K = (4/3)\,[(5/4-\nu)\beta + (1-\nu^2)\beta^2]. \tag{3.4}$$

In (3.4), it should be noted that $\dot{\psi} = \psi$ since $\dot{P} = P$. Differentiation of the equation of radial equilibrium (2.9) with respect to t and substitution using (3.1) – (3.4) lead to

$$[(1+4\beta)/K]\,\dot{W}'''' + 4\psi\,\dot{W}'' + [4/(1+4\beta)]\,\dot{W} = 0 \tag{3.5}$$

which holds at the instant $t = 0$. In (3.5) and all following expressions, primes denote differentiation with respect to ξ. In obtaining (3.5), use is made of the condition that $w = W = 0$ at $t = 0$.

For an infinitely long shell, a solution of (3.5) can be taken in the form

$$\dot{W} \propto \cos\alpha\xi. \tag{3.6}$$

Then (3.5) is satisfied provided

$$4\psi = 4(1+4\beta)^{-1}\alpha^{-2} + (1+4\beta)K^{-1}\alpha^2. \tag{3.7}$$

The minimum value of ψ, for $0 \leqslant \alpha < \infty$, is $K^{-1/2}$. Thus the minimum load for bifurcation is determined by the condition

$$\psi K^{1/2} = 1 \tag{3.8}$$

and bifurcation cannot occur when

$$\psi K^{1/2} < 1. \tag{3.9}$$

In view of (2.5) – (2.8), the plastic strains go to zero in the limit as $H \to \infty$. It can easily be verified using the definitions in (3.3), (3.4) that the bifurcation condition (3.8) reduces to the well-known elementary result for elastic buckling [2] in the limit as $H \to \infty$, namely

$$P = P_{cr} = (Eh/a)\,[3\,(1-\nu^2)]^{-1/2}. \tag{3.10}$$

On the other hand, the rigid-plastic case can be obtained from (3.8) along with (3.3), (3.4) in the limit as $E \to \infty$. As $E \to \infty$, it follows that $K \to 0$ and $\psi, P \to \infty$, so bifurcation cannot occur.

4. Perturbation analysis

When N_x, κ_θ are neglected in the constitutive relations (2.3) – (2.8), these relations can be reduced to,

$$(1 + 4\beta)\dot{N}_\theta + 3N_\theta = 4Hh\dot{e}_\theta, \tag{4.1}$$

$$K\ddot{M}_x + (1 + \beta)\dot{M}_x = (Hh^3/36)\,[(1 + 4\beta)\ddot{\kappa}_x + 3\,\dot{\kappa}_x]. \tag{4.2}$$

It is of interest to note that, in the limiting case of a rigid-plastic material, the coefficient K appearing with the highest derivative on the left side of (4.2) vanishes. This difference strongly affects the method of solution in the rigid-plastic case considered in [1] and the elastic-plastic case considered here. In either case, the constitutive relation between M_x and κ_x can be integrated once with respect to t. This integration introduces an arbitrary function of x into the expression for M_x. From (2.2), (2.9), (3.3), (3.4), (4.1) and (4.2), it follows that W satisfies the equation

$$\begin{aligned}
&[(1 + 4\beta)^2\,\ddot{W} + 6\,(1 + 4\beta)\,\dot{W} + 9\,W]''''\\
&+ 4\,\psi\,[K\,(1 + 4\beta)\,\dddot{W} + (1 + \beta)\,(1 + 4\beta)\,\ddot{W} + K\,(5 + 8\beta)\,\ddot{W} + 4\,(1 + \beta)^2\,W\\
&+ 4K\,(1 + \beta)\,W]'' + 4K\,\dddot{W} + 4\,(1 + \beta)\,\ddot{W} = Q\,(\xi).
\end{aligned} \tag{4.3}$$

It is recalled that primes denote differentiation with respect to ξ and $\dot{\psi} = \psi$. The function $Q\,(\xi)$ on the right side of (4.3) depends on the initial values of M_x and W and their derivatives. $Q\,(\xi)$ can be determined directly in terms of W by putting $t = 0$ on the left side of (4.3). Equation (4.3) is inhomogeneous if either $W(\xi, 0)$, $\dot{W}(\xi, 0)$ or $\ddot{W}(\xi, 0) \neq 0$, and these three functions can be specified arbitrarily as initial conditions on (4.3).

A solution to (4.3) is now constructed by introducing an infinite series representation for W,

$$W(\xi, t) = \sum_{n=0}^{\infty} [\psi(t)]^n W_n(\xi, t). \tag{4.4}$$

The initial conditions to be specified on W will be even functions in ξ. Hence $W(\xi, t)$ can be represented by a Fourier cosine transform $A(\alpha, t)$, where the definition of the transform as given by Sneddon [4] is used. Then, with $A_n(\alpha, t)$ denoting the Fourier cosine transform of $W_n(\xi, t)$ and $R(\alpha)$ the transform of $Q(\xi)$, equations (4.3) and (4.4) become

$$\begin{aligned}&[(1+4\beta)^2\dddot{A} + 6(1+4\beta)\ddot{A} + 9\dot{A}]\alpha^4 \\ &-4\psi[K(1+4\beta)\ddot{A} + (1+\beta)(1+4\beta)\dot{A} + K(5+8\beta)\dot{A} + 4(1+\beta)^2 A \\ &+4K(1+\beta)A]\alpha^2 + 4K\ddot{A} + 4(1+\beta)\dot{A} = R\end{aligned} \tag{4.5}$$

and

$$A = \sum_{n=0}^{\infty} \psi^n A_n. \tag{4.6}$$

The infinite series (4.6) is substituted for A on the left side of (4.5). Equating coefficients of ψ^0 on both sides of (4.5) yields an equation for determining $A_0(\alpha, t)$,

$$[(1+4\beta)^2\ddot{A}_0 + 6(1+4\beta)\dot{A}_0 + 9A_0]\alpha^4 + 4K\ddot{A}_0 + 4(1+\beta)\dot{A}_0 = R. \tag{4.7}$$

The general solution of (4.7) for $A_0(\alpha, t)$ can be written

$$A_0 = G_0 e^{rt} + K_0 e^{st} + R_0 \tag{4.8}$$

where

$$r = -(c-\gamma)/q \leqslant 0, \quad s = -(c+\gamma)/q < 0, \quad 0 \leqslant \alpha < \infty \tag{4.9}$$

and

$$c = 2(1+\beta) + 3(1+4\beta)\alpha^4, \quad q = 4K + (1+4\beta)^2\alpha^4, \tag{4.10}$$

$$\gamma = 2\{(1+\beta)^2 + [3(1+\beta)(1+4\beta) - 9K]\alpha^4\}^{1/2} \tag{4.11}$$

and G_0, K_0, R_0 are arbitrary functions of α. This general solution for $A_0(\alpha, t)$ contains three arbitrary functions needed to meet the initial conditions on (4.3). An infinite system of second order ordinary differential equations with constant coefficients for A_n, $n \geqslant 1$, can be obtained by equating to zero coefficients of ψ^n on the left side of (4.5) when the infinite series (4.6) is substituted for A. Since A_0 already introduces three arbitrary functions for meeting the initial conditions on (4.3), the initial conditions

$$A_n(\alpha, 0) = \dot{A}_n(\alpha, 0) = 0, \quad n \geqslant 1 \tag{4.12}$$

can be assigned. Laplace transforms [4] in the variable t of the functions $A_n(\alpha, t)$ are introduced and denoted by $\bar{A}_n(\alpha, p)$. In terms of these Laplace transforms, this system of equations can be written, for $n = 1$,

$$\{[(1+4\beta)(p+1)+3]^2\alpha^4 + 4(p+1)[K(p+1)+(1+\beta)]\}\bar{A}_1$$

$$= 4\alpha^2\{(1+\beta)(1+4\beta)[p\bar{A}_0 - A_0(\alpha, 0)] + 4(1+\beta)^2\bar{A}_0\}$$

$$+ 4K\alpha^2\{(1+4\beta)[p^2\bar{A}_0 - pA_0(\alpha, 0) - \dot{A}_0(\alpha, 0)]$$

$$+ (5+8\beta)[p\bar{A}_0 - A_0(\alpha, 0)] + 4(1+\beta)\bar{A}_0\} \tag{4.13}$$

and for $n \geqslant 2$,

$$\bar{A}_n(\alpha, p) = f_n(\alpha, p)\bar{A}_{n-1}(\alpha, p) \tag{4.14}$$

where

$$f_n(\alpha, p) = \frac{4[K(p+n)+(1+\beta)][(1+4\beta)(p+n)+3]\alpha^2}{[(1+4\beta)(p+n)+3]^2\alpha^4 + 4(p+n)[K(p+n)+(1+\beta)]}.$$

In the rigid-plastic case, $K = 0$, so $f_n(\alpha, p)$ is $O(p^{-1})$ as $p \to \infty$. Then $f_n(\alpha, p)$ is, in itself, a Laplace transform, and a solution of (4.14) for $A_n(\alpha, t)$ can be written as a convolution integral depending on $A_{n-1}(\alpha, t)$. In the present case, this procedure does not hold.

5. Solution for initial geometric imperfection

Obtaining a solution to the original partial differential equation (4.3) is reduced to determining W_0, W_1 and $W_n (n \geqslant 2)$ from (4.7), (4.13) and (4.14) respectively for specified initial conditions. From (4.6), (4.12) it follows that

$$A_0(\alpha, 0) = A(\alpha, 0), \quad \dot{A}_0(\alpha, 0) = \dot{A}(\alpha, 0) \tag{5.1}$$

while $\ddot{A}_0(\alpha, 0)$ can be determined by first setting $t = 0$ on the left side of (4.5); thus R is determined in terms of $A(\alpha, 0)$, $\dot{A}(\alpha, 0)$ and $\ddot{A}(\alpha, 0)$. With R known, setting $t = 0$ on the left side of (4.7), and using (5.1) determines $\ddot{A}_0(\alpha, 0)$ in terms of $A(\alpha, 0)$, $\dot{A}(\alpha, 0)$, $\ddot{A}(\alpha, 0)$. With $A_0(\alpha, 0)$, $\dot{A}_0(\alpha, 0)$, $\ddot{A}_0(\alpha, 0)$ determined, the three arbitrary functions $G_0(\alpha)$, $K_0(\alpha)$, $R_0(\alpha)$ in (4.8) can be found, and $A_0(\alpha, t)$ is completely determined. Three distinct initial-value problems can be considered by taking, in turn, one of the three functions $A(\alpha, 0)$, $\dot{A}(\alpha, 0)$, $\ddot{A}(\alpha, 0)$ to be nonzero with the other two identically zero. The same technique to be presented applies in the three cases, so only one case, $A(\alpha, 0) \neq 0$ with $\dot{A}(\alpha, 0) \equiv \ddot{A}(\alpha, 0) \equiv 0$ is treated. In the moderately thick shells considered, little buckling deformation occurs until loading is well into the plastic state. The datum $t = 0$ can be chosen arbitrarily at any instant in the loading history once the specimen has reached the fully plastic state by simply using the current value of the uniform compressive stress P as the value for P_0 in the relation $P = P_0 e^t$. By taking $t = 0$ at a time before the buckling deformation becomes large, $W(\xi, 0)$ and its Fourier cosine transform $A(\alpha, 0)$ can be identified with a geometric imperfection in the unloaded specimen. For small t, the analysis then applies to the early stages of plastic buckling when the buckling mode becomes established.

The method of steepest descents [5] is employed in inverting the Fourier cosine transform $A_0(\alpha, t)$ to obtain $W_0(\alpha, t)$, so it is convenient to represent the initial imperfection by

$$W(\xi, 0) = e^{-b\xi^2}. \tag{5.2}$$

Thus

$$A(\alpha, 0) = B = (2b)^{-1/2} e^{-\alpha^2/4b}. \tag{5.3}$$

Specification of the initial conditions is completed by putting $\dot{W}(\xi, 0) = \ddot{W}(\xi, 0) = 0$, and hence

$$\dot{A}(\alpha, 0) = \ddot{A}(\alpha, 0) = 0 . \tag{5.4}$$

For b large, (5.2) describes an initial imperfection localized to the neighborhood of the origin. In view of (5.1) and (5.4)

$$\dot{A}_0(\alpha, 0) = 0, \tag{5.5}$$

and (4.5), (4.7) yield

$$\ddot{A}_0(\alpha, 0) = -16\psi_0(1+\beta)(1+\beta+K)[4K + (1+4\beta)^2\alpha^4]^{-1}\alpha^2 B. \tag{5.6}$$

where, from (3.4),

$$\psi_0 = \psi(0) = 3P_0 a/(2Hh).$$

For the initial conditions (5.3), (5.5), (5.6), equation (4.8) takes the form

$$A_0(\alpha, t) = B\{1 - C_0\alpha^{-2}[1 - e^{-ct/q}((c/\gamma)\sinh(\gamma t/q) + \cosh(\gamma t/q))]\} \tag{5.7}$$

where

$$C_0 = (16/9)(1+\beta)(1+\beta+K)\psi_0 .$$

Since the right side of (5.7) is an even function in γ, $A_0(\alpha, t)$ is a single-valued function of α. $A_0(\alpha, t)$ has a pole of order two at the origin and essential singularities where $q = 0$, that is, where

$$\alpha = (\pm 1 \pm i)K^{1/4}(1+4\beta)^{-1/2}. \tag{5.8}$$

$A_0(\alpha, t)$ has the Laurent's series representation

$$A_0(\alpha, t) = BL_0(\alpha, t) \tag{5.9}$$

where $L_0(\alpha, t)$ is the Laurent's series beginning

$$L_0(\alpha, t) = 1 - C_0\alpha^{-2}[1 - e^{-3t/(1+4\beta)}(1 + 3t/(1+4\beta))] + O(\alpha^{-6}). \tag{5.10}$$

The Laurent's series $L_0(\alpha, t)$ converges for

$$|\alpha| > \delta_0 = \sqrt{2}K^{1/4}(1+4\beta)^{-1/2} \tag{5.11}$$

where δ_0 is the radius of the circle centered at the origin that passes through the essential singularities located by (5.8). From (5.3), (5.9) the inversion integral in the Fourier cosine transformation for determining $W_0(\alpha, t)$ has the form, in terms of a new variable $\zeta = b\alpha$,

$$W_0(\xi, t) = (b/4\pi)^{1/2}\int e^{b(-\zeta^2/4+i\xi\zeta)}L_0(b\zeta, t)\,d\zeta. \tag{5.12}$$

This integral has the same form as in the rigid-plastic analysis [1], and can be treated in the same way, as follows. The path of integration is taken as the real axis indented at the origin by a semi-circle Γ of radius $\delta > \delta_0/b$ in the upper half-plane. As the semi-circle Γ is traversed in the clockwise direction, $\zeta = -\delta e^{-i\phi}$, with ϕ increasing from zero to π. Hence, along Γ,

$$|e^{b(-\zeta^2/4+i\xi\zeta)}| \leqslant e^{b\delta^2/4}e^{-\xi b\delta\sin\phi}, \quad 0 \leqslant \phi \leqslant \pi.$$

The integrand in (5.12) can be made to go exponentially to zero along Γ as $\delta \to 0$ by requiring that $b\delta \to \infty$ and $b\delta^2 \to 0$. Putting $\delta = \delta_0 b^{-2/3}$ and letting $b \to \infty$ satisfies both of these requirements. It should also be noted that $\delta = \delta_0 b^{-2/3} > \delta_0 b^{-1}$ as $b \to \infty$, so the semi-circle Γ is exterior to the circle of convergence $|\zeta| = \delta_0/b$ of the Laurent's series L_0 in the ζ plane. Also, when $b\delta \to \infty$, $L_0 \to 1$. The essential singularities do not contribute to the inversion integral, at least not in the limit as $b \to \infty$. Accordingly, the path of integration along the real axis in the inverse Fourier transformation can be deformed into any path lying in the upper half-plane, along which the order of summation and integration can be interchanged on the right side of (5.12) since L_0 converges everywhere along the path. The method of steepest descents can now be applied in the term-by-term integration of the right side of (5.12). The factor $\exp[b(-\zeta^2/4 + i\xi\zeta)]$ is common to all integrands in the series, and so all terms have the same saddle-point, $\zeta = 2\xi i$, and paths of steepest descent which are straight lines parallel to the real axis through the saddle-point. Since the leading term of L_0 in (5.10) is unity, the leading term in the asymptotic expansion of $W_0(\xi, t)$ as $b \to \infty$ is simply

$$W_0(\xi, t) \sim e^{-b\xi^2} = W(\xi, 0) \tag{5.13}$$

and thus the first term in (4.4) is the initial imperfection.

With $W_0(\xi, t)$ given by (5.13),

$$A_0(\alpha, t) = A(\alpha, 0) = B \tag{5.14}$$

and $A_1(\alpha, t)$ satisfying the initial conditions (4.12) can be obtained from (4.13) as

$$A_1(\alpha, t) = C_1\{1 - e^{-t(1+c/q)}[\cosh(\gamma t/q) + (c+q)\gamma^{-1}\sinh(\gamma t/q)]\} \tag{5.15}$$

where

$$C_1 = \frac{4(1+\beta)(1+\beta+K)\alpha^2 B}{(1+\beta+K)+4(1+\beta)^2\alpha^4}.$$

The expression in curly brackets in (5.15) is a single-valued function of α with essential singularities at the locations given by (5.8). This expression has the Laurent's series representation $L_1(\alpha, t)$ which begins

$$L_1(\alpha, t) = 1 - e^{-t}e^{-3t/(1+4\beta)}[1 + 4(1+\beta)(1+4\beta)^{-1}t + O(\alpha^{-4})] \tag{5.16}$$

and converges for $|\alpha| > \delta_0$. For $t \ll 1$,

$$L_1(\alpha, t) = 8(1+\beta)^2(1+4\beta)^{-2}t^2 + O(\alpha^{-4}). \tag{5.17}$$

When just the leading term in L_1 is retained, which determines the leading term in the asymptotic expansion of W_1 for large b, A_1 can be written

$$A_1(\alpha, t) = BCt^2 \tag{5.18}$$

where

$$C = C(\alpha) = \frac{32(1+\beta)^3(1+\beta+K)\alpha^2}{(1+4\beta)^2[(1+\beta+K)+4(1+\beta)^2\alpha^4]}. \tag{5.19}$$

It can be noted that $A_1(\alpha, t)$ as given by (5.18) satisfies the initial conditions (4.12). Then, from (5.18),

$$\bar{A}_1(\alpha, p) = 2BCp^{-3} \tag{5.20}$$

The coefficient $f_n(\alpha, p)$ in the recursion relation (4.14) is a rational function of p with simple poles in the complex p plane where

$$p = (-n+r), \quad (-n+s)$$

and r, s are given by (4.9). For real α, these poles lie on the negative real axis in the p plane. The inverse Laplace transform of $\bar{A}_n(\alpha, p)$ in powers of t is obtained by expressing the right side of (4.14) as a Laurent's series in powers of p^{-1}, which is convergent for p sufficiently large. For $p \to \infty$, $f_n(\alpha, p)$ is written

$$f_n(\alpha, p) = \rho + O(p^{-1}) \tag{5.21}$$

where

$$\rho = \frac{4K(1+4\beta)\alpha^2}{(1+4\beta)^2\alpha^4 + 4K} \leqslant K^{1/2}, \quad -\infty < \alpha < \infty. \tag{5.22}$$

The leading term ρ on the right side of (5.21) is independent of n. The leading term of $A_2(\alpha, t)$ in powers of t is obtained by substituting ρ for f_n, and the right side of (5.20) for $\bar{A}_1$, on the right side of (4.14). Thus

$$A_2(\alpha, t) = t^2 BC\rho \tag{5.23}$$

and continuing the procedure gives

$$A_n(\alpha, t) = t^2 BC\rho^{n-1}, \quad n \geqslant 1. \tag{5.24}$$

Then, substitution in (4.6) from (5.14), (5.24) yields

$$A(\alpha, t) = A(\alpha, 0)\,[1 + t^2\,\psi C(1 + \psi\rho + \ldots + \psi^n\rho^n + \ldots)]. \tag{5.25}$$

With $S(\alpha)$ representing the sum

$$S(\alpha) = \sum_{n=0}^{\infty} \psi^n \rho^n = (1 - \psi\rho)^{-1} = \frac{4K + (1+4\beta)^2\alpha^4}{4K - 4K\psi(1+4\beta)\alpha^2 + (1+4\beta)^2\alpha^4} \tag{5.26}$$

equation (5.25) can be re-written

$$A(\alpha, t) = A(\alpha, 0)\,[1 + t^2\,\psi C(\alpha) S(\alpha)]. \tag{5.27}$$

In view of the bifurcation condition (3.8), (3.9) and the inequality $\rho \leqslant K^{1/2}$ in (5.22), the series $S(\alpha)$ converges uniformly in α, $-\infty < \alpha < \infty$, provided the load parameter ψ is less than the value for bifurcation. The expression $C(\alpha)S(\alpha)$ in (5.27) is a rational function of α with simple poles. Its inverse Fourier cosine transform $G(\xi)$ can easily be found by means of the residue theorem. Hence, for $\xi \geqslant 0$,

$$G(\xi) = iN[(G_1 e^{i\alpha_1\xi} - G_1^* e^{-i\alpha_1^*\xi}) + (G_2 e^{i\alpha_2\xi} - G_2^* e^{-i\alpha_2^*\xi})], \tag{5.28}$$

where

$$N = 8(2\pi)^{1/2}(1+\beta)^3(1+\beta+K)(1+4\beta)^{-2}, \tag{5.29}$$

$$\alpha_1 = (1+4\beta)^{-1/2}K^{1/4}[(1+\psi K^{1/2})^{1/2} + i(1-\psi K^{1/2})^{1/2}], \tag{5.30}$$

$$G_1 = \frac{-i2\psi K^{1/2}(1+\beta)\alpha_1^3}{(1+4\beta)(1-\psi^2 K)^{1/2}\,[1+\beta+K+4(1+\beta)^2\alpha_1^4)]}, \tag{5.31}$$

$$\alpha_2 = (1+i)(1+\beta+K)^{1/4}(1+\beta)^{-1/2}/2, \tag{5.32}$$

$$G_2 = \frac{4K + (1+4\beta)^2\alpha_2^4}{4(1+\beta)^2[4K - 4K\psi(1+\beta)\alpha_2^2 + (1+4\beta)^2\alpha_2^4]\,\alpha_2}, \tag{5.33}$$

and the asterisk * denotes complex conjugates. The terms in α_1 arise from the residues at the poles of $S(\alpha)$, while the terms in α_2 arise from the residues at the poles of $C(\alpha)$. Finally, $W(\xi, t)$ can be obtained from (5.27), (5.28) by using the convolution theorem for the Fourier cosine transform [4],

$$W(\xi, t) = e^{-b\xi^2} + (2\pi)^{-1/2} t^2 \psi \int_0^\infty e^{-b\eta^2} [G(\xi+\eta) + G(|\xi-\eta|)] d\eta \tag{5.34}$$

where $\xi \geqslant 0$. Laplace's method [5], applied to the integral in (5.34) yields, as $b \to \infty$,

$$W(\xi, t) = e^{-b\xi^2} + (2b)^{-1/2} t^2 \psi G(\xi) \tag{5.35}$$

and it is recalled that ξ, ψ are defined by (3.3), (3.4).

In examining the behavior of $G(\xi)$, it is useful to consider two limiting cases: (i) the purely elastic case obtained in the limit as $H \to \infty$; and (ii) the rigid-plastic case obtained in the limit as $E \to \infty$.

(i) As $H \to \infty$, from (3.3), (3.4), the following orders of magnitude are noted:

$$\beta = O(H), \quad K = O(H^2), \quad \psi = O(H^{-1}), \tag{5.36}$$

and, with P_{cr} defined by (3.10) and P representing the current value of the uniform compressive stress,

$$\psi K^{1/2} = P/P_{cr} = O(1). \tag{5.37}$$

Then, in view of (5.29) – (5.32) and (5.36), (5.37),

$$N = O(H^3), \quad \alpha_1 = O(1), \quad \alpha_2 = O(1), \quad G_1 = O(H^{-2}) \tag{5.38}$$

In obtaining the order relation for G_2, it needs to be noted that the terms of leading order $O(H^2)$ in the numerator of (5.33) cancel, with the result that

$$G_2 = O(H^{-3}). \tag{5.39}$$

Hence,

$$\psi N G_1 = O(1), \quad \psi N G_2 = O(H^{-1}) \to 0. \tag{5.40}$$

Thus the terms in α_2 drop out in this limit. The amplitude of the buckling deformation is governed by the factor

$$(1 - P^2/P_{cr}^2)^{-1/2} \exp[-k\xi(1 - P/P_r)^{1/2}] \tag{5.41}$$

where

$$k\xi = [(1-\nu^2)/12]^{1/4} \xi = [3(1-\nu^2)]^{1/4} (ah)^{-1/2} x$$

which agrees with results for elastic buckling in [2]. For $P/P_{cr} < 1$, the deformation dies out exponentially with distance x from the site of the imperfection. As $P/P_{cr} \to 1$,

$$(1 - P^2/P_{cr}^2)^{-1/2} \to \infty, \quad \exp[-k\xi(1 - P/P_{cr})^{1/2}] \to 1.$$

Thus the deformation spreads farther and farther along the shell as the critical load is approached and the amplitude increases indefinitely.

(ii) In the rigid-plastic limit obtained as $E \to \infty$, equations (3.3), (3.4) give β, $K \to 0$, and hence $G_1 \to 0$. Then (5.28) reduces to

$$G(\xi) = -8\sqrt{\pi}\, e^{-\xi/2} \sin(\xi/2 - \pi/4) \tag{5.42}$$

which, apart from a multiplicative constant, is the function $F_1(\xi)$ obtained in the rigid-plastic analysis in [1].

For the general elastic-plastic case, in view of (3.8), it is convenient to define ψ_{cr} such that

$$\psi_{cr} K^{1/2} = 1 \tag{5.43}$$

and to put

$$\lambda = \psi/\psi_{cr} = \psi K^{1/2}. \tag{5.44}$$

For many ductile metals, including the 6061-T6 aluminium alloy used in the specimens in [1],

$$\beta = H/E \ll 1$$

for loading well past the yield point, and it is then appropriate to neglect β compared to unity. Then

$$iN(G_1 e^{i\alpha_1 \xi} - G_1^* e^{-i\alpha_1^* \xi})$$

$$= -64(2\pi)^{1/2} K^{3/4} \lambda (1 - \lambda^2)^{-1/2} \exp[-K^{1/4}(1-\lambda)^{1/2}\xi] \cdot$$

$$\sin[K^{1/4}(1+\lambda)^{1/2}\xi + \phi] \tag{5.46}$$

where

$$\tan\phi = \frac{1-2\lambda}{1+2\lambda}\left(\frac{1+\lambda}{1-\lambda}\right)^{1/2}$$

and K is given approximately by

$$K = (5 - 4\nu)\beta/3. \tag{5.47}$$

When β is neglected compared to unity, the terms in α_2 coincide with the limiting form as $E \to \infty$, namely,

$$iN(G_2 e^{i\alpha_2 \xi} - iG_2^* e^{-i\alpha_2^* \xi}) = -8\sqrt{\pi}\, e^{-\xi/2} \sin(\xi/2 - \pi/4). \tag{5.48}$$

6. Comparison with experiment

Typical values for the material properties E, H, ν for 6061-T6 aluminium alloy are

$$E = 73\ GPa, \quad H = 0.70\ GPa, \quad \nu = 0.33$$

which give

$$K = 0.012.$$

The critical compressive stress P_{cr} for bifurcation is given by (3.4) and (3.8) as, with $a/h = 8$ from [1],

$$P_{cr} = 0.53\ GPa.$$

The average of the maximum axial loads obtained in the experiments in [1] corresponds to a uniform compressive stress

$$P = 0.36\ GPa$$

and hence

$$\lambda = 0.68$$

For these numerical values, (5.46) becomes

$$iN(G_1 e^{i\alpha_1 \xi} - G_1^* e^{-i\alpha_1^* \xi})$$

$$= -5.4 \exp(-0.19\,\xi) \sin(0.43\,\xi - 0.34) \tag{6.1}$$

and, for comparison, (5.48) can be written

$$iN(G_2 e^{i\alpha_2 \xi} - G_2^* e^{-i\alpha_2 \xi})$$

$$= -14.2 \exp(-0.5\,\xi) \sin(0.5\,\xi - 0.78). \tag{6.2}$$

The right side of (6.2) represents the deformation obtained using rigid-plastic analysis, while the right side of (6.1) is the additional deformation resulting from consideration of elastic strains. At $\xi = 0$, where the radial displacement is a maximum, the right side of (6.1) is 15 per cent of the right side of (6.2). Hence the intuitive expectation that the elastic strains make only a small contribution to buckling deformation when buckling occurs well past the yield point is corrorborated.

7. Discussion

The present analysis shows that the bifurcation load is an upper bound on the buckling load, but does not establish a means of predicting what the maximum load will be. The analysis does show that, since the experimentally determined maximum load is considerably less than the load at bifurcation, the buckling deformation dies out exponentially with distance from the site of the imperfection. If buckling were governed by the bifurcation condition, that is, by $\lambda \to 1$, the exponential damping factor on the right side of (5.46) would approach unity, and the buckling deformation should show a strong tendency to spread. Plastic deformation is history-dependent. Bifurcation analysis neglects the effects of initial imperfections or boundary constraints on the growth of buckling deformation, and thus does not account for the localization of buckling deformation observed in experiments.

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